For You To Do Solution: Sets and Relations

William Deng

- Specifying using comprehension notation
  - Odd positive integers: $\{ x : \mathbb{N} \mid \exists y (y \in \mathbb{N} \text{ and } x = 2y + 1) \}$. Or equivalently, $\{ 2k + 1 \mid k \in \mathbb{N} \}$.
  - The squares of integers: $\left\{ x : \mathbb{Z} \mid \exists y (y \in \mathbb{Z} \text{ and } x = y^2) \right\}$. Or equivalently, $\left\{ x^2 \mid x \in \mathbb{Z} \right\}$
- Express the following logic properties on sets without using the # operator. Assume the set is S.
  - Set has at least one element: $\exists x (x \in S)$. Or equivalently, $S \neq \emptyset$.
  - Set has no elements: $\forall x (x \notin S)$. Or equivalently, $S = \emptyset$.
  - Set has exactly one element: $\exists x (x \in S \text{ and } \forall y (y \in S \implies y = x))$. Or equivalently, $\exists x (x \in S \text{ and } S \setminus \{x\} = \emptyset)$.
  - Set has at least two elements: $\exists x (\exists y (\{x, y\} \subseteq S \text{ and } x \neq y))$. Or equivalently, $\exists x (x \in S \text{ and } S \setminus \{x\} \neq \emptyset)$.
  - Set has exactly two elements: $\exists x (\exists y (\{x, y\} \subseteq S \text{ and } x \neq y \text{ and } \forall z (z \in S \implies z = x \text{ or } z = y)))$. Or equivalently, $\exists x (\exists y (\{x, y\} \subseteq S \text{ and } x \neq y \text{ and } S \setminus \{x, y\} = \emptyset))$.
- Express the following properties of pairs of sets
  - Two sets are disjoint. Let assume the two sets are S_1 and S_2. Answer: $S_1 \cap S_2 = \emptyset$.
  - Two sets form a partitioning of a third set. Let assume that S_1 and S_2 form a partitioning of S_3. Answer: $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S_3$.
- Which of the following are functions?
  - Parent = {(John,Autumn),(John,Sam)}. No, Parent is not a function because Parent maps John to Autumn and Sam.
  - Square = {(1,1),(-1,1),(-2,4)}. Yes.
  - ClassGrades = {(Todd,A),(Virg,B)}. Yes.
- What kind of function/relation is Abs?
  Abs = $\{ (x, y) : \mathbb{Z} \times \mathbb{N} \mid (x < 0 \text{ and } y = -x) \text{ or } (x >= 0 \text{ and } y = x) \}$
  Abs is a total function and surjective.

- What kind of function/relation is Squares?
  Squares : $\mathbb{Z} \times \mathbb{N}$, Squares = {(-1,1),(2,4)}
  Squares is a partial function and is one-to-one.

- What operators ($\cap, \cup, \setminus$) preserve function-ness if an operator fails to preserve a property give an example.
  - $\cap$ yes;
  - $\cup$ no. Let $S = \{1, 2\}$ and $f, g : S \times S$ and $f = \{(1, 1)\}, g = \{(1, 2)\}$. Then $f \cup g$ ={(1,1),(1,2)}.
  - $\setminus$ yes;

- What operators ($\cap, \cup, \setminus$) preserve onto-ness if an operator fails to preserve a property give an example.
  - $\cap$ no. Let $S = \{1, 2\}$ and $f, g : S \times S$ and $f = \{(1, 1), (2, 2)\}, g = \{(1, 2), (2, 1)\}$. Then $f \cap g = \emptyset$.
  - $\cup$ no. Let f, g be the ones defined above. Then $f \cup g$ is not a function.
  - $\setminus$ no. Take any $f = g$. Then $f \setminus g = \emptyset$.

- What operators ($\cap, \cup, \setminus$) preserve 1-1-ness if an operator fails to preserve a property give an example.
  - $\cap$ yes;
  - $\cup$ no, let $S = \{1, 2\}$ and $f, g : S \times S$ and $f = \{(1, 1), (2, 2)\}, g = \{(1, 2), (2, 1)\}$. Then $f \cup g$ ={(1,1),(1,2), (2,2),(2,1)} which is not 1-to-1.
  - $\setminus$ yes;

- What operators, composition (;), closure (+), transpose (˜) preserve function-ness if an operator fails to preserve a property give an example.
  - composition (;) yes;
  - closure (+) no; Let $S = \{1, 2\}$ and $f : S \times S$ and $f = \{(1, 2), (2, 1)\}$. Then $+f = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ which is not a function.
  - transpose (˜) no; Let $S = \{1, 2\}$ and $f : S \times S$ and $f = \{(1, 1), (2, 1)\}$.

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- What operators, composition (;), closure (+), transpose (˜) preserve 1-1-ness if an operator fails to preserve a property give an example.

– composition (;) yes;
– closure (+) no; Let $S = \{1, 2\}$ and $f : S \times S$ and $f = \{(1, 2), (2, 1)\}$. Then $+f = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ which is not a one-to-one.
– transpose (˜) yes.